6-2017

# Locally Compact Groups: Traditions and Trends

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K.H. Hofmann 86

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$\leftarrow$ Gordon Thomas Whyburn $\leftarrow$ Robert Lee Moore

# Locally compact groups: Traditions and Trends

Karl Heinrich Hofmann,
TU Darmstadt, Germany
& Tulane University,
New Orleans, USA.

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# Landmarks

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1934 Lev S. Pontryagin
1937 Egbert van Kampen
Duality of Locally Compact Abelian Groups

# Center of Century

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1940, 1953, 1965, 1979

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Then one knows:
*Every locally compact almost connected group $G$ is approximated by Lie group quotients $G/N$.* (Projective limit)

# The Second Half of the 20th Century

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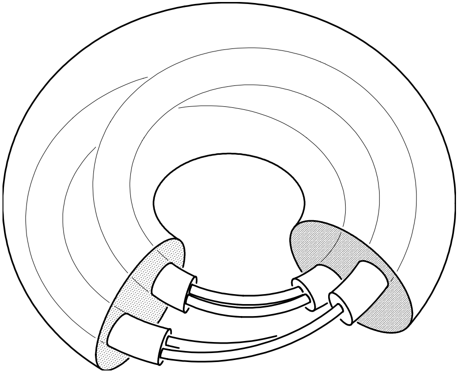

# Solenoid

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$G$ is far from a Lie group, yet it has the Lie algebra $\mathbb{R}$ and its exponential function $\exp\colon \mathbb{R} \to G$ is a group morphism.

The assignment $G \mapsto \mathfrak{g}$ of a Lie algebra $\mathfrak{L}(G) = \mathfrak{g}$ to a locally compact group $G$ is functorial.

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If $f\colon G \to H$ is a morphism of topological groups, then

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is a commuting diagram.

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is a commuting diagram.

exp *is a natural transformation.*

# Dimension

The underlying topological vector space of the Lie algebra $\mathfrak{g}$ of a locally compact group is $\cong \mathbb{R}^d$ for a cardinal $d$.

The vector spaces $V = \mathbb{R}^{(d)}$ and $W = \mathbb{R}^d$ are dual to each other, i.e.

$$\mathrm{Hom}(V, \mathbb{R}) \cong \mathbb{R}^d \qquad \text{and} \qquad \mathrm{Hom}_{cont}(W, \mathbb{R}) = \mathbb{R}^{(d)}.$$

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The cardinal $d$ is assigned canonically to any locally compact group $G$ and is called the *dimension* $\dim G$ of $G$.

Every cardinal occurs even in the class of compact groups: Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the circle group, then $\dim \mathbb{T}^d = d$.

# Topological dimension of locally compact groups

**Theorem**

*On locally compact groups $G$ all concepts of topological dimensions agree and equal* $\dim G$.

# Zero-Dimensional Groups

## Corollary

*A locally compact group is zero-dimensional if and only if it is totally disconnected iff its Lie algebra is singleton.*

# Historic roots of zero-dimensional locally compact groups

—Galois groups (of algebraic extensions of fields)

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—nonarchimedian completions of $\mathbb{Q}$ yield the $p$-adic fields $\mathbb{Q}_p$, their linear algebra, eventually the Lie group theory over $\mathbb{Q}_p$.

—also the nonarchimedean ($p$-adic) completion of $\mathbb{Z}$, called $\mathbb{Z}_p$.

# One glance back

Lie theory $[G \mapsto \mathfrak{L}(G)]$ introduces algebra into the topological structure theory of topological groups:
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cf. components, arc components, direct decompositions, covering homomorphisms, transformation groups etc.

Zero-dimensionality nixes the influence of Lie theory on topology: other branches of mathematics must enter: number theory, finite group theory, $p$-adic Lie algebra theory. Abelian locally compact group theory without $\mathbb{R}^m$ and the torus $\mathbb{T}^n$.

# Zero-dimensional locally compact groups: Some historic landmarks

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# Grand Literature on dim-0 groups

Generic and monographic literature on noncompact zero-dimensional locally compact groups:

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**NONE as of yet**

# Hyperspaces

For a compact space $X$ the set of closed subsets is a compact space called its *hyperspace* $\mathcal{H}(X)$. The function

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[See Leopold Vietoris: Stetige Mengen, Monatshefte **31** (1921)]
[L. Vietoris, 1891—2002.]

# The Chabauty space of a locally compact group

**Definition.** For a locally compact group $G$ let $\mathcal{SUB}(G)$ denote the set of all closed subgroups (with the topology of the hyperspace of $G \cup \{\infty\}$ (and the set $\mathcal{SUB}(G)$ embedded into it) is called the *Chabauty space* of $G$. The function

$$g \mapsto \overline{\langle g \rangle} : G \to \mathcal{SUB}(G)$$

—is it even continuous? –A natural question!

[see Claude Chabauty: Limit d'ensemble et géométrie des nombres, Bull. Soc. Math. France **78** (1950)]

# "Nostrification"

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The Chabauty space was rediscovered by Bourbaki in the context of Haar measure theory as the space of idempotents in the measure algebra $M(G)$ (uncomplicated only if $G$ is compact).

[see Nicolas Bourbaki, Intégration, Chap. 7 et 8, Hermann, Paris, 1963]

# Some results and examples

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[Hofmann and Willis, 2015]

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**Example.** *If $G = \mathbb{R}$ then*

$$\mathcal{SUB}(\mathbb{R}) = \{r{\cdot}\mathbb{Z} : 0 \leq r\} \cup \mathbb{R} \textit{ homeom } [0,1].$$

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*Inside $\mathcal{SUB}(\mathbb{R})$*

Note: *$G$ is approximated by discrete subgroups $\cong \mathbb{Z}$ inside $\mathcal{SUB}(G)$*

**Next Example.** *If $G = \mathbb{Z}_p$, then*

$$\mathcal{SUB}(G) = \{p^n \cdot \mathbb{Z}_p : n{=}0, 1, 2, \dots\} \cup \{\{0\}\}$$

*homeomorphic to $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.*

# A quite horrible example: The circle group

**Further Example.** *If $G = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, then*

$$g \mapsto \overline{\langle g \rangle} : \mathbb{T} \to \mathcal{SUB}(\mathbb{T}) = \left\{ \frac{\frac{1}{n} \cdot \mathbb{Z}}{\mathbb{Z}} : n \in \mathbb{N} \right\} \cup \{\{\mathbb{T}\}\}$$

$$\textit{homeom } \{1, 2, 3, \ldots, \infty\}$$

*is surjective; the set of all "irrational" points in $\mathbb{T}$ is dense and maps onto $\mathbb{T} \leftrightarrow \infty$. The function $g \mapsto \overline{\langle g \rangle}$ is continuous at all "irrational" points and discontinuous at all "rational" ones.*

# Inductively monothetic groups

**Definition.** A subgroup $H$ of a topological group is called *monothetic* if it is closed and has a dense cyclic subgroup. If is called *inductively monothetic* if it is closed and any closed subgroup having a dense finitely generated subgroup is monothetic.

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• $\mathbb{T}^2$ (compact 2-torus) is monothetic but not inductively monothetic. (It contains the 4 element subgroup $\left\{\frac{1}{2}\cdot\mathbb{Z}/\mathbb{Z}\right\}^2$ which is not monothetic.)

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- $\mathbb{Q}$ (discrete) is inductively monothetic but not monothetic.
- $\mathbb{T}^2$ (compact 2-torus) is monothetic but not inductively monothetic. (It contains the 4 element subgroup $\left\{\frac{1}{2}\cdot\mathbb{Z}/\mathbb{Z}\right\}^2$ which is not monothetic.)
- However: Fact. All 0-*dimensional* monothetic groups are inductively monothetic.

# Classification of Inductively Monothetic Groups

## Theorem

[Herfort,Hofmann,Russo] *A locally compact group $G$ is inductively monothetic if and only if one of the following conditions is satisfied:*

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(1) *$G$ is a one-dimensional monothetic group.*

(2) *$G$ is discrete and isomorphic to a subgroup of $\mathbb{Q}$.*

(3) *$G$ is isomorphic to a local product*

$$\prod_{p \,\mathrm{prime}}^{\mathrm{loc}} (G_p, C_p),$$

*where $G_p$ is either $\cong \mathbb{Z}(p^n)$, $n = 0, 1, \ldots, \infty$, or $\mathbb{Z}_p$ or $\mathbb{Q}_p$.*

Here: $\prod_{j\in J}^{\mathrm{loc}}(G_j, C_j)$ is a subgroup of $\prod_{j\in J} G_j$ containing all $(g_j)_{j\in J}$ such that $\{j \in J : g_j \notin C_j\}$ is finite.

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The topology on the local product is obtained by declaring $\prod_{j\in J} C_j$ with its product topology open. For infinite $J$ it is finer than the topology induced from $\prod_{j\in J} G_j$.

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The class of inductive monothetic groups is called $\mathcal{IM}$.

# Selfduality of the class $\mathcal{IM}$

**Corollary**

*The class $\mathcal{IM}$ is closed under passage to the character group.*

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Note: This property makes the class $\mathcal{IM}$ formally quite different from the class of monothetic groups.

# Groups $G$ approximable by groups $\cong \mathbb{Z}$ in $\mathcal{SUB}(G)$

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### Theorem

*Let $G$ be a locally compact group. Then the following statements are equivalent:*

(1) *In $\mathcal{SUB}(G)$ we have $G \in \overline{\{H \in \mathcal{SUB}(G) : H \cong \mathbb{Z}\}}$.*

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*Let $G$ be a locally compact group. Then the following statements are equivalent:*

(1) *In $\mathcal{SUB}(G)$ we have $G \in \overline{\{H \in \mathcal{SUB}(G) : H \cong \mathbb{Z}\}}$.*

(2) *Either $G \cong \mathbb{R} \times \mathrm{comp}(G)$ and $G/G_0$ is inductively monothetic, or else $G$ is $\cong$ to a subgroup of $\mathbb{Q}$.*

[Hamrouni, Hofmann]

# The Definition of Near Abelian Groups

## Definition

A topological group will be called *near abelian* if it is locally compact and and there is a closed normal subgroup $A$ such that

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(2) Every closed subgroup of $A$ is normal in $G$.

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There is a representation $\psi\colon G \to \mathrm{Aut}\, A$, $\psi(g)(a) = gag^{-1}$.

$\ker \psi$
$= C_G(A) = \{g \in G : (\forall a \in A : ag = ga\}$,
is the centralizer of $A$ in $G$.

# The centralizer $C_G(A)$

Clearly, $A \subseteq C_G(A)$;
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If $\psi(A) \subseteq \{\mathrm{id}, -\mathrm{id}\}$, then $G$ is said to be $A$-*trivial*.

# First Results

## Theorem

*Let $G$ be an $A$-nontrivial near abelian group. Then*

(1) *$A$ is periodic (i.e. $A$ totally disconnected and $A \subseteq \mathrm{comp}(A)$.*

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(3) *$G$ is periodic iff $G/A$ is periodic iff $G/A \ncong$ a subgroup of $\mathbb{Q}$.*

This explains why in the context of near abelian groups the theory of *periodic groups* is significant.

# Factorisation and Scaling

**Definition**

Let $G$ be a near abelian group with base $A$. A closed subgroup $H$ of $G$ is called a *scaling subgroup* if

(i) $H$ is inductively monothetic, and

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If $H$ satisfies (i) and
(ii′) $G = C_G(a)H$,
then $H$ is called a *small* scaling subgroup.

# A Note on the Significance of Scaling

If $G = AH$,
since $H$ is $\sigma$-compact and has an open compact subgroup,
there is a quotient homomorphism

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The product $AH$ is not too far from the semidirect product $A \rtimes H$.

# A Fundamental Theorem

## Theorem

*Let $G$ be a periodic near abelian group with base $A$ such that $G$ is $A$-nontrivial. Then there is a small scaling subgroup, i.e.*
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*Moreover, $H = \prod^{loc}_{p \text{ prime}}(G_p, C_p)$ and $G_p$ is either a finite cyclic $p$-group or a $p$-adic group $\cong \mathbb{Z}_p$.*

[Herfort, Hofmann, Russo]

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We call such groups $H$ $\Pi$-*procyclic*. These groups are not necessarily compact.

# An Alternative Fact (Theorem)

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**Theorem**

*Let $G$ be a locally compact group and $A$ any* compact *normal subgroup such that $G/A$ is isomorphis to a discrete subgroup of $\mathbb{Q}$. Then there is a closed subgroup $H$ such that $G = A \rtimes H$.*

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It is unknown to which extent these facts remain valid if the the compactness of $A$ is relaxed to closedness.

# Scalar Automorphisms

**Lemma**

*Let $G$ be a locally compact abelian periodic group. and $\alpha$ a continuous automorphism. Then the following statements are equivalent:*

(1) $\alpha(H) \subseteq H$ *for every closed subgroup* $H \leq G$.

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(1) $\alpha(H) \subseteq H$ *for every closed subgroup* $H \leq G$.

(2) $\alpha(\overline{\langle g\rangle}) \subseteq \langle g\rangle$ *for every* $g \in G$.

(3) $\alpha(g) \in \langle g\rangle$ *for every* $g \in G$.

# Scalar Automorphisms

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(3) $\alpha(g) \in \langle g \rangle$ *for every* $g \in G$.

(4) *There is an* $r \in \widetilde{\mathbb{Z}}$ *such that* $\alpha(g) = r{\cdot}g$ *for all* $g \in G$.

Here
$\widetilde{\mathbb{Z}} = \prod_{p \text{ prime}} \mathbb{Z}_p$ is the universal profinite compactification of $\mathbb{Z}$.
Its (discrete) character group is $\cong \mathbb{Q}/\mathbb{Z}$.

The subgroup of $\mathrm{Aut}\, G$ containing all scalar automorphisms is written $\mathrm{SAut}\, G$.

The Lemma shows that the representation

$$r \mapsto \{x \mapsto r\cdot x\} : (\widetilde{Z})^{\times} \to \mathrm{SAut}\, G$$

is surjective for each locally compact abelian periodic group $G$. (For any ring $R$ let $R^{\times}$ denote the group of units, i.e. invertible elements.)

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For each near abelian group $G$ with base $A$ and periodic $G/A$ we have a representation $G/A \to \mathrm{SAut}\, A$ therefore $\mathrm{SAut}\, A$ and so $(\widetilde{\mathbb{Z}})^{\times}$ must be understood.

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Each profinite (=compact 0-dimensional) Abelians group $G$ is the product $\prod_{p \text{ prime}} G_p$ of its $p$-Sylow subgroups $G_p$ (also called $p$-primary components).

# The Structure of $\mathbb{Z}_p^\times$

Let $p$ be an odd prime. Then the $p$-Sylow subgroup of $\mathbb{Z}_P^\times$ contains $1 + p\mathbb{Z}_p$ (=the image of $p\mathbb{Z}_p$ under the exponential function).

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$$x \mapsto \exp p{\cdot}x : \mathbb{Z}_p \to (1 + p{\cdot}Z_p, \times) \subseteq \mathbb{Z}_p^\times$$

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Also, $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ is the field of $p$ elements, and its multiplicative group of units is cyclic of order $p-1$.

This group lifts to $\mathbb{Z}_p^\times$, and in the end we have

$$(\forall p > 2) \quad \mathbb{Z}_p^\times \cong \mathbb{Z}_p \times \mathbb{Z}(p-1) \text{ (additively written).}$$

The exceptional case is $p = 2$:

$$\mathbb{Z}_2^\times \cong \mathbb{Z}_2 \times \mathbb{Z}(2) \text{ (additively written).}$$

# Decomposing $\widetilde{\mathbb{Z}}^{\times}$

We know $\widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, therefore,

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where

$$q - 1 = p_1^{k_1} \cdots p_n^{k_n}, \text{ with } n = n(q) \text{ and } p_j | (q-1).$$

At this point one loses track without some simple graph theory.

# The Mastergraph

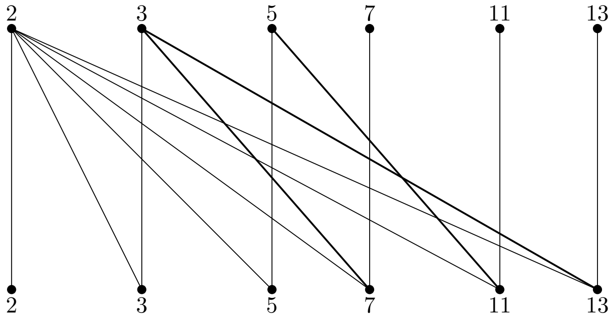


FIGURE 1. The initial part of the master graph.

# Graph theoretical interpretation for $\widetilde{\mathbb{Z}}^{\times}$

We have a sloping edge $e = ((m,1),(n,0))$ with $m < n$ (that is, prime $p_m \leftrightarrow (m,1)$ in the top row and prime $q_n \leftrightarrow (n,0)$ in the bottom row) iff $p_m | (q_n - 1)$.

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$\mathbb{Z}_{q_n}^{\times} \cong \mathbb{Z}_{q_n} \times \mathbb{Z}(q_n - 1)$,
and so we have a cyclic $p_m$- group $\mathbb{S}_e \cong \mathbb{Z}(p_m^{k_e})$ for a suitable natural number $k_e$. For the vertical edge $e$ coming down to $q_n$ write $\mathbb{S}_e = \mathbb{Z}_{q_n}$.

# Structure of $\mathbb{Z}_q^\times$

Let $\mathcal{F}_q$ denote the finite set of sloping edges ending up in the lower vertex $q = q_n \leftrightarrow (n, 0)$
and $\mathcal{E}_p$ the set of all edges coming down from $p = p_m \leftrightarrow (m, 1)$.

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Then

$$\mathbb{Z}_q^\times = \prod_{e \in \mathcal{F}_q} \mathbb{S}_e = \mathbb{Z}_q \times \prod_{e \in \mathcal{F}_q,\ \mathrm{s}loping} \mathbb{Z}(p_e^{k(e)}).$$

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Remember $\widetilde{\mathbb{Z}}^\times = \prod_{p\ \mathrm{p}rime} \widetilde{\mathbb{Z}}_p^\times$

where $\widetilde{\mathbb{Z}}_p^\times = \prod_{e \in \mathcal{E}_p} \mathbb{S}_e$
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Thus $\widetilde{\mathbb{Z}}^\times$ is the product of the procyclic $p$-groups $\mathbb{Z}_e$:

$$\widetilde{\mathbb{Z}}^\times = \prod_e \mathbb{S}_e$$

where $e$ ranges through ALL edges of the mastergraph.

# The Master-Graph of a Near Abelian Group

If $G$ is near abelian with base $A$, then we define the *graph* $\mathcal{G}$ *of* $G$ as a subgraph of the master-graph of $\widetilde{\mathbb{Z}}$

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Then $\mathcal{G}$ gives considerable insight into the structure of $G$.

# The Benefit from the use of Graphs

The examples show that the use of even technically simple graphs can help in the organization of an almost impenetrable Sylow subgroup structure of locally compact periodic groups such as $\widetilde{\mathbb{Z}}$ and, more generally of all near abelian locally compact groups.

# Summarizing Applications

This presentation is a preview of a monograph by
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(C) Strongly topologically quasihamiltonian groups $G$
$[(\forall A, B \leq G \text{ closed}) \quad AB = \overline{AB} \leq G]$.